

## Graph Theory Group Work

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A car rental company has three locations in Mexico City: the International Airport, Oficina Vallejo, and Downtown. Customers can drop off their vehicles at any of these locations. Based on prior experience, the company expects that, at the end of each day, 40% of the cars that begin the day at the Airport will end up Downtown, 50% will return to the Airport, and 10% will be at Oficina Vallejo. Similarly, 60% of the Oficina Vallejo cars will end up Downtown, with 30% returning to Oficina Vallejo and 10% at the airport. Finally, 30% of the Downtown cars will end up at each of the other locations, with 40% staying at the Downtown location.

**Question.** Model this situation with a directed network. If the company starts with all the cars at the Airport, how will the cars be distributed after two days of rentals?

**Further Question.** If all the cars start at the Downtown location, approximately how will the cars be distributed after a year? In a year, would the fact that they all started at the Downtown location make a difference?

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Let  $G = (V, E)$  be a simple, connected, undirected graph, and let

$$P = \{\{v_1, v_2\} \mid v_1, v_2 \in V, v_1 \neq v_2\}$$

be the set of all unordered pairs of vertices. Define a function  $f : E \rightarrow P$  as follows: If  $e \in E$  is an edge in  $G$ , then  $f(e) = \{a, b\}$ , where  $a$  and  $b$  are the vertices that  $e$  touches.

**Question.**

- (a) Explain why  $f(e)$  is always a set of size 2; i.e. show that  $f$  is well-defined.
- (b) Is  $f$  one-to-one? Prove or disprove. If not, when is it one-to-one?
- (c) Is  $f$  onto? Prove or disprove. If not, when is it onto?
- (d) How would this construction change if you allowed for directed graphs?

**Further Question.** Can you think of a scenario where it would be desirable for each edge to have more than 2 vertices that it touches (say, for instance, 3 vertices)? Generalize the construction above to account for this scenario.

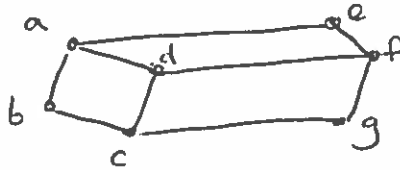
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Let  $G = (V, E)$  be a connected, undirected graph. Define a relation  $R$  on  $V$  as follows: for any vertices  $a, b \in V$ ,  $a R b$  if there is a path from  $a$  to  $b$  with an even number of edges. (A path may use the same edge twice)

**Question.**

- Prove that  $R$  is an equivalence relation.
- Find the equivalence classes for the graph below.
- What if a path could not use the same edge twice. Would  $R$  still be an equivalence relation? Prove or disprove.



**Further Question.** If the graph were allowed to be directed, would  $R$  still be an equivalence relation? What if it were unconnected? Can you think of a scenario where this type of equivalence relation could be used?

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Let  $G = (V, E)$  be a graph. A *subgraph*,  $H$ , of  $G$  is a graph with vertex set  $V_H \subseteq V$  and edge set  $E_H \subseteq E$ .

**Question.**

- (a) Write down all the subsets of  $K_3$ , the complete graph on three vertices.
- (b) Let  $V_H \subseteq V$  and  $E_H \subseteq E$ . Is there always a subgraph of  $G$  with vertex set  $V_H$  and edge set  $E_H$ ? Explain.
- (c) Let  $G$  be a connected, undirected graph. Prove that there is a subgraph  $T$  of  $G$  such that  $T$  contains all the vertices of  $G$ , and  $T$  is a tree. (Such a subgraph is called a *spanning tree*.) Give a constructive proof that explains how to construct a spanning tree of a graph.

**Further Question.** Is there more than one spanning tree? Can you say exactly how many there will be?

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Suppose you are organizing a soccer league which has ten teams, each with a different home field. You want to schedule each team for six games (3 home and 3 away) with each game being against a different opponent.

**Question.** Is this possible? Explain how you would model it (if it is possible) and give a model if it is possible. Explain. How many possible configurations are there?

**Further Question.** Suppose, in addition, that you want to make each team travel the same amount (or as close to as possible). How would you change your model? Suppose that you also want each team to play four games on Saturday and only two on Sunday due to time constraints. How would you change your model?